Property (T), algebraic characterizations and higher variants

Talk 1:

Motivation: Property (T) can be defined in many equiv ways

Most of them are of "analytic nature" $\sim$ related to unitary reps

eg: (Γ, S) fin gen gp $S = S^{-1}$

Γ has (T) iff $\exists \lambda > 0$ $\forall \pi : \Gamma \to \mathcal{U}(\mathcal{H})$ unitary rep of Γ

$$\pi(\Delta^2 - \lambda\Delta) \geq 0$$

where $\Delta = 1 - \frac{1}{|S|} \sum_{s \in S} s \in \mathbb{R}\Gamma$

$$\left(\pi(\Delta) = Id - \frac{1}{|S|} \sum_{s \in S} \pi(s) \in \mathcal{B}(\mathcal{H}) \right)$$

Ozawa: Γ has (T) iff $\exists \lambda > 0$ $\exists \xi_i \in \mathbb{R}\Gamma$ st
(fin many)

$$\Delta^2 - \lambda\Delta = \sum_i \xi_i^* \xi_i$$

$\rightsquigarrow$ $Aut(F_n)$ has property (T) for $n \geq 4$
(Kaluba - Kielak - Nowak) $n \geq 6$
(Kaluba - Nowak - Ozawa) $n = 5$
(Nitsche) $n = 4$

Goal: Explore Ozawa-like characterizations for higher variants of (T).

Group cohomology:

$H^1(\Gamma, \pi)$ Hausdorff $\forall \pi$ unirep $\iff$ $H^1(\Gamma, \pi) = 0$ $\forall \pi$ unirep

$\uparrow$ Delorme-Guichardet thm

$H^n(\Gamma, \pi)$ Hausdorff $\forall \pi$ unirep $\Leftarrow$ / $\not\Rightarrow$ $H^n(\Gamma, \pi) = 0$ $\forall \pi$ unirep

$\Updownarrow$ $\qquad$ $\Updownarrow$ (Bader - Nowak)

$\exists \lambda > 0$, $\forall \pi : \Gamma \to \mathcal{U}(\mathcal{H})$ $\pi((\Delta_{n-1}^+)^2 - \lambda \Delta_{n-1}^+) \geq 0$ $\qquad$ $\exists \lambda > 0$ $\exists M_i \in M_k(\mathbb{R}\Gamma)$ $\Delta_n - \lambda I = \sum M_i^* M_i$

$\not\Leftrightarrow$ $\qquad$ $\Leftrightarrow$ (sometimes)

$\exists \lambda > 0$, $\exists M_i \in M_k(\mathbb{R}\Gamma)$

$$(\Delta_{n-1}^+)^2 - \lambda \Delta_{n-1}^+ = \sum M_i^* M_i$$

$\Delta_n^+ = d_n^* d_n$

$\Delta_n^- = d_{n-1} d_{n-1}^*$ $\qquad$ $\Delta_n = \Delta_n^- + \Delta_n^+$

Laplacian $\quad$ Lapian $\quad$ Lacian

Today: I (T) vs (FH) and group cohomology

II Functions conditionally of negative type

III Delorme - Guichardet

I.

Def: A topological gp G has Property (T) if $\exists Q \subset G$ compact $\exists \varepsilon > 0$ st $\forall \pi : G \to \mathcal{U}(\mathcal{H})$ continuous unirep, $\mathcal{H}^{\pi(G)} = 0$ ($\{v \in \mathcal{H} \; \pi(g)v = v\}$)

we have: $\sup_{g \in Q} \| \pi(g)\xi - \xi \| \geq \varepsilon \quad \forall \xi \in \mathcal{H} \; \|\xi\| = 1$

Introduced by Kazhdan in 1967

- Higher rank simple Lie gps w finite center have (T)
- (T) passes down to lattices
- (T) + loc compact $\Rightarrow$ compactly generated.

Def: A top gp has Property (FH) if $\forall$ continuous isometric action of G on a Hilb space $\mathcal{H}$ has a fixed point.

Prop: For G loc compact σ-compact,

$$(FH) \Rightarrow (T)$$

Thm (Delorme Guichardet) For G top gp, we have

$$(T) \Rightarrow (FH)$$

1) Group cohomology

From now on, G will be loc compact second countable (lcsc)

Data: G lcsc gp
$+ \ \rho : G \to O(V)$ continuous isometric rep of G on a Banach V

$$C^k(G,V) = C(G^{k+1}, V)$$

$$C^k(G,V)^G = \{ c : G^{k+1} \to V \text{ cont st } c(gg_0,\dots,gg_k) = \rho(g)c(g_0,\dots,g_k) \ \forall g, g_0 \dots g_k \in G \}$$

These spaces are endowed with the top of uniform convergence on compact subsets
↳ Fréchet spaces.

$$d_k : C^k(G,V) \to C^{k+1}(G,V)$$

$$(d_k c)(g_0,\dots,g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(g_0,\dots,\hat{g}_i,\dots,g_{k+1})$$

exercise: Show that $d_{k+1} \circ d_k = 0$
and that $d_k(C^k(G,V)^G) \subset C^{k+1}(G,V)^G$

$$0 \to C^0(G,V)^G \xrightarrow{d} C^1(G,V)^G \xrightarrow{d} C^2(G,V)^G \to \dots$$

Def: $$H^k_{ct}(G,\rho,V) := \frac{\ker(d_k \mid C^k(G,V)^G)}{\operatorname{Im}(d_{k-1} \mid C^{k-1}(G,V)^G)} \quad k \geq 1$$
↳ top vector space (not necessarily Hausdorff)

$$\overline{H}^k_{ct}(G,\rho,V) = \frac{\ker(d_k \mid C^k(G,V)^G)}{\overline{\operatorname{Im}(d_{k-1} \mid C^{k-1}(G,V)^G)}} \quad k \geq 1$$
↳ Hausdorff (in fact Fréchet)

Low degree cohom

- $k=0$ $\quad C^0(G,V)^G = C(G,V)^G \simeq V$, $\quad \{g \mapsto \rho(g)v\} \leftarrow v$

$$(d_0 c)(g_0,g_1) = c(g_1) - c(g_0)$$

$c \in \ker d_0 \cap C^0(G,V)^G \iff c(g) = \rho(g)v$ with $v \in V^{\rho(G)}$

$$\boxed{H^0_{ct}(G,\rho) = V^{\rho(G)}}$$

- $k=1$ $\quad C^1(G,V)^G = C(G^2,V)^G \simeq C(G,V)$

$$c \longmapsto \{g \mapsto c(1,g) = \tilde{c}(g)\}$$

$$\left(c(g,h) = \rho(g)c(1,g^{-1}h) = \rho(g)\tilde{c}(g^{-1}h) \right)$$

$$(d_1 c)(g_0,g_1,g_2) = c(g_1,g_2) - c(g_0,g_2) + c(g_0,g_1)$$
$$= \rho(g_1)\tilde{c}(g_1^{-1}g_2) - \rho(g_0)\tilde{c}(g_0^{-1}g_2) + \rho(g_0)\tilde{c}(g_0^{-1}g_1)$$

$$(d_1 c) = 0 \iff c(g_0^{-1}g_2) = \rho(g_0^{-1}g_1)\tilde{c}(g_1^{-1}g_2) + \tilde{c}(g_0^{-1}g_1)$$

$g_0^{-1}g_1 = g$
$g_1^{-1}g_2 = h$ $\quad \boxed{c(gh) = c(g) + \rho(g)c(h)}$

Under the same identification

$$\begin{array}{ccc} C^0(G,V)^G & \xrightarrow{d_0} & C^1(G,V)^G \\ \cong & & \cong \\ V & \longrightarrow & C(G,V) \\ v & \longmapsto & \{g \mapsto \rho(g)v - v\} \end{array}$$

We obtain:

$$H^1_{ct}(G,\rho) = \frac{\overbrace{\{c : G \to V \text{ cont } c(gh) = c(g) + \rho(g)c(h)\}}^{Z^1(G,\rho)}}{\underbrace{\{g \mapsto \pi(g)v - v, \ v \in V\}}_{B^1(G,\rho)}}$$

2) Relation to (FH)

Cont affine isometric action of G on a Banach V

$\alpha : G \to \operatorname{Isom}(V)$ $\quad \alpha(g)v = \underbrace{\rho(g)}_{\text{linear part}}v + \underbrace{b(g)}_{\text{translation part}}$

$\alpha(gh) = \alpha(g)\alpha(h)$ $\quad \rho : G \to O(V)$ cont isom rep

$\Updownarrow$

$b(gh) = b(g) + \rho(g)b(h)$
i.e $b \in Z^1(G,\rho)$

When does an affine action have fixed pts?

$\alpha(g)v = v \ \forall g \in G \iff b(g) = v - \rho(g)v \ \forall g \in G$

We proved:

Prop: $\forall$ isom rep (ρ, V) we have.

$H^1_{ct}(G,\rho) = 0 \iff$ Every cont affine isom action of G on V with linear part ρ has a fixed point

In particular

G has (FH) $\iff H^1_{ct}(G,\pi) = 0 \ \forall \pi$ continuous unitary rep of G

3) Relation with (T)

Prop: Given G lcsc gp, $(\pi, \mathcal{H})$ unitary rep of G on a Hilbert sp $\mathcal{H}$ wo inv vectors

$H^1_{ct}(G,\pi)$ Hausdorff (i.e d_0 has a closed image) $\iff$ $\forall Q \subset G$ compact, $\exists \varepsilon > 0$ $\sup_{g\in Q} \|v - \pi(g)v\| \geq \varepsilon$ $\forall v \in \mathcal{H}$ $\|v\| = 1$

Proof: Open mapping theorem

$d : \mathcal{H} \to Z^1(G, \mathcal{H})$ is a cont linear map
$v \longmapsto \{g \mapsto v - \pi(g)v\}$

It has closed image iff $\forall Q \subset G$ compact $\exists \varepsilon > 0$
$\|v\| = 1$, $\|dv\|_Q = \sup_{g\in Q} \|v - \pi(g)v\| < \varepsilon \Rightarrow v \in \ker d$

$v \in \ker d \iff v \in \mathcal{H}^{\pi(G)} = 0.$ $\square$

Recap: For G lcsc gp.

- (T) $\iff$ $H^1_{ct}(G,\pi)$ Hausdorff $\forall \pi$ cont uni rep
- (FH) $\iff$ $H^1_{ct}(G,\pi) = 0$ $\forall \pi$ cont uni rep.
- In particular (FH) $\Rightarrow$ (T).

II - Functions conditionally of negative type and Schoenberg's thm.

1) Cocycle GNS

Def: G top gp. A continuous fct $\varphi : G \to \mathbb{R}$ is <u>conditionally of negative type</u> if:
- $\varphi(e) = 0$
- $\varphi(g^{-1}) = \varphi(g)$ $\forall g \in G$
- $\forall n \in \mathbb{N}$ $\forall g_1, \dots, g_n \in G$, $\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$ st $\sum_{i=1}^n \lambda_i = 0$
$$\sum_{i,j=1}^n \lambda_i \lambda_j \varphi(g_i^{-1} g_j) \leq 0$$

Thm (Cocycle GNS) Gelfand - Naimark - Segal

- For every fct $\varphi : G \to \mathbb{R}$ cond of neg type, $\exists \pi : G \to \mathcal{U}(\mathcal{H})$ a cont uni rep of G and $b \in Z^1(G,\pi)$
$$\varphi(g) = \|b(g)\|^2 \quad \forall g \in G$$
- Conversely $\forall \pi : G \to \mathcal{U}(\mathcal{H})$ cont uni rep and $b \in Z^1(G,\pi)$ the fct $g \mapsto \|b(g)\|^2$ is cond of neg type.

Proof: $\varphi : G \to \mathbb{R}$ cond of neg type.

$V = \{ f : G \to \mathbb{R}$ finitely supported fcts $\sum_{x\in G} f(x) = 0 \}$

φ defines an inner product on V

$$\langle f, f' \rangle_\varphi = -\frac{1}{2} \sum_{g,h\in G} f(g) f'(h) \varphi(g^{-1}h)$$

$N_\varphi = \{ f \in V, \langle f, f \rangle_\varphi = 0 \}$ lin subspace by Cauchy-Schwarz

V/N_φ positive definite inner product

$\mathcal{H}$ = Hilb space completion of $(V/N_\varphi, \langle \cdot, \cdot \rangle_\varphi)$

$G \curvearrowright V$ by left translation $(g\cdot f)(x) = f(g^{-1}x)$

$$\langle g\cdot f, g\cdot f' \rangle_\varphi = \langle f, f' \rangle_\varphi.$$

$G \curvearrowright \mathcal{H}$ after quotienting and completing

Let $b(g) = \delta_g - \delta_e$

$$\|b(g)\|^2 = \langle \delta_g - \delta_e, \delta_g - \delta_e \rangle_\varphi$$
$$= \underbrace{\langle \delta_g, \delta_g \rangle_\varphi}_{= -\frac{1}{2}\cdot 2\varphi(e) = 0} + \underbrace{\langle \delta_e, \delta_e \rangle_\varphi}_{= -\frac{1}{2}\cdot 2\varphi(e) = 0} - 2\langle \delta_e, \delta_g \rangle_\varphi$$
$$= -2\cdot\left(-\frac{1}{2}\right) \varphi(g) = \varphi(g).$$

Conversely, if π unirep of G and $b \in Z^1(G,\pi)$

$g_1, \dots, g_n \in G$ $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ $\sum \lambda_i = 0$

$$\boxed{\|b(g^{-1}h)\|^2} = \|b(g^{-1}) + \pi(g^{-1})b(h)\|^2$$
$$= \|\underbrace{\pi(g)b(g^{-1})}_{=-b(g)} + b(h)\|^2$$
$$= \|b(h) - b(g)\|^2$$
$$= \boxed{\|b(g)\|^2 + \|b(h)\|^2 - 2\langle b(g), b(h) \rangle}$$

$$\sum_{i,j} \lambda_i \lambda_j \|b(g_i^{-1}g_j)\|^2 = \sum_{i,j} \lambda_i \lambda_j \left(\cancel{\|b(g_i)\|^2} + \cancel{\|b(g_j)\|^2} - 2\langle b(g_i), b(g_j) \rangle \right)$$

$\sum_{i,j} \lambda_i \lambda_j \|b(g_i)\|^2 = \underbrace{\left(\sum_j \lambda_j\right)}_{=0} \sum_i \lambda_i \|b(g_i)\|^2$

$$= -2 \sum_{i,j} \lambda_i \lambda_j \langle b(g_i), b(g_j) \rangle$$
$$= -2 \left\| \sum \lambda_i b(g_i) \right\|^2 \leq 0$$
$\square$

Schoenberg's thm: If $\varphi : G \to \mathbb{R}$ is a fct cond of neg type then $\forall t > 0$ $e^{-t\varphi}$ is of <u>positive type</u>

i.e. $\forall n \in \mathbb{N}$ $\forall g_1, \dots, g_n \in G$ $\forall \lambda_1, \dots, \lambda_n \in \mathbb{R}$ (no condition on λ)
$$\sum_{i,j=1}^n \lambda_i \lambda_j e^{-t\varphi(g_i^{-1}g_j)} \geq 0$$

Proof: $\varphi(g) = \|b(g)\|^2$ for some π and b. , $\boxed{t = 1}$

$$-\varphi(g^{-1}h) + \varphi(g) + \varphi(h) = 2\langle b(g), b(h) \rangle$$

$\sum \lambda_i \lambda_j e^{-\varphi(g_i^{-1}g_j)}$
$$= \sum \lambda_i \lambda_j e^{2\langle b(g_i), b(g_j) \rangle} e^{-\varphi(g_i)} e^{-\varphi(g_j)}$$
$$= \sum_{i,j} \lambda_i \lambda_j \sum_{k\geq 0} \frac{2^k}{k!} \langle b(g_i), b(g_j) \rangle^k e^{-\varphi(g_i)} e^{-\varphi(g_j)}$$
$$= \sum_{k\geq 0} \frac{2^k}{k!} \sum_{i,j} \lambda_i \lambda_j \langle b(g_i)^{\otimes k}, b(g_j)^{\otimes k} \rangle e^{-\varphi(g_i)} e^{-\varphi(g_j)}$$
$$= \sum_{k\geq 0} \frac{2^k}{k!} \left\| \sum_i \lambda_i e^{-\varphi(g_i)} b(g_i)^{\otimes k} \right\|^2_{\mathcal{H}^{\otimes k}} \geq 0$$

$\sum \mathcal{H}^{\otimes k}$ $\leftarrow$ Fock space

Cor: (Slowing down cocycles)

If $\varphi : G \to \mathbb{R}$ is a fct of cond neg type, then $\forall 0 < \alpha < 1$, φ^α is also cond of neg type

Proof: We have a Lévy-Khintchine rep for $\mathbb{R}_+ \to \mathbb{R}_+$, $x \mapsto x^\alpha$

$x > 0$
$$x^\alpha = \frac{\alpha}{\Gamma(\alpha)} \int_0^\infty (1 - e^{-tx}) t^{-\alpha-1} dt$$

(to prove this, differentiate wrt x and change variables)

φ is cond of neg type $\overset{\text{Schoenberg}}{\rightsquigarrow}$ $e^{-t\varphi}$ is of positive type
$\rightsquigarrow$ $1 - e^{-t\varphi}$ is again cond of neg type

$\sum \lambda_i \lambda_j \varphi(g_i^{-1}g_j)^\alpha$
$$= \frac{\alpha}{\Gamma(\alpha)} \int_0^\infty \underbrace{\sum_{i,j} \lambda_i \lambda_j (1 - e^{-t\varphi(g_i^{-1}g_j)})}_{\leq 0} t^{-\alpha-1} dt$$
$\leq 0.$

$\square$